Chapter 1

Calculation Objects

GMAT has the ability to calculate numerous quantities that are dependent the states of objects, coordinate systems, and the mission sequence. These calculation objects can range from the spacecraft state, to the local atmospheric density, to the positions of celestial bodies with respect to spacecraft, or other celestial bodies. In this section, we present how GMAT performs these calculations by showing the mathematical algorithms.

1.1 Spacecraft State Representations

There are several state representations that can be used in GMAT to define the state of a spacecraft object. These include the Keplerian elements, Cartesian state, Equinoctial elements, Spherical Elements, and the modified Keplerian elements. In the following few subsections we discuss the definitions of these states types, and show how GMAT converts between the different state representations.

1.1.1 Definitions

The Keplerian Elements are one of the most commonly used state representations. They provide a way to define the spacecraft state in way that provides an intuitive understanding of the motion of the spacecraft in orbit. The Keplerian elements are denoted a, e, i, ω, Ω, and ν. They are defined in detail in Table 1.1 and illustrated in Fig. 1.1. Sections 1.1.2 and 1.1.3 show the algorithm that GMAT uses to convert from Keplerian elements to the cartesian state respectively.

The cartesian state is another common state representation and is often used in the numerical integration of the equations of motion. The cartesian state with respect to a given coordinate system is described in detail in Table 1.2.

The equinoctial elements are a set of non-singular elements that can be used to describe the state of a spacecraft. Because they are nonsingular, they are useful for in expressing the equations of motion in Variation of Parameters (VOP) form. The elements can be unintuitive to use however. The equinoctial elements are described in detail in Table 1.3.

The modified Keplerian Elements are similar to the Keplerian elements except a and e are replaced with the radius of periapsis r_p, and the radius of apoapsis r_a. r_p and r_a are often more convenient and intuitive for describing the dimensions of a Keplerian orbit than a and e. The modified Keplerian Elements are defined in detail in Table 1.5.

1.1.2 Cartesian State to Keplerian State

Given: $\mathbf{r}$, $\mathbf{v}$, and μ
Find: a, e, i, ω, Ω, and ν

First calculate the specific angular momentum and its magnitude.

$$\mathbf{h} = \mathbf{r} \times \mathbf{v} \tag{1.1}$$

$$h = \|\mathbf{h}\| \tag{1.2}$$

$$\hat{\mathbf{h}} = \frac{\mathbf{h}}{h} \tag{1.3}$$

Next, calculate the orbital parameter, p, the radius magnitude, r, and the speed, v.

$$p = \frac{h^2}{\mu} \tag{1.4}$$

$$r = \|\mathbf{r}\| \tag{1.5}$$

$$v = \|\mathbf{v}\| \tag{1.6}$$

Using the above quantities, the semimajor axis, eccentricity, and inclination are calculated using the following three equations:

$$a = \frac{r}{2 - \dfrac{rv^2}{\mu}}; \tag{1.7}$$

Table 1.1: The Keplerian Elements

Symbol	Name	Description
a	semimajor axis	The semimajor contains information on the type and size of an orbit. If $a > 0$ the orbit is elliptic. If $a < 0$ the orbit is hyperbolic.
e	eccentricity	The eccentricity contains information on the shape of an orbit. If $e = 0$, then the orbit is circular. If $0 < e < 1$ the orbit is elliptical. If $e = 1$ the orbit is parabolic. If $e > 1$ then the orbit is hyperbolic.
i	inclination	The inclination is the angle between the $\hat{\mathbf{z}}_I$ axis and the orbit normal direction $\mathbf{h}$. If $i \leq 90°$ then the orbit is prograde. If $i > 90°$ then the orbit is retrograde.
ω	argument of periapsis	The argument of periapsis is the angle between a vector pointing at periapsis, $\mathbf{x}_p$, and a vector pointing at the spacecraft. The argument of periapsis is undefined for circular orbits.
Ω	right ascension of the ascending node	Ω is defined as the angle between $\hat{\mathbf{x}}_I$ and $\mathbf{N}$ measured counterclockwise. $\mathbf{N}$ is defined as the vector pointing from the center of the central body to the spacecraft, when the spacecraft crosses the bodies equatorial plane from the southern to the northern hemisphere. Ω is undefined for equatorial orbits.
ν	true anomaly	The true anomaly is defined as the angle between a vector pointing at periapsis, $\mathbf{x}_p$, and a vector pointing at the spacecraft. The true anomaly is undefined for circular orbits.

Table 1.2: The Cartesian State

Symbol	Description
x	x-component of position
y	y-component of position
z	z-component of position
$\dot{x}$	x-component of velocity
$\dot{y}$	y-component of velocity
$\dot{z}$	z-component of velocity

Table 1.3: The Equinoctial Elements

Symbol	Description
a	The semimajor contains information on the type and size of an orbit. If $a > 0$ the orbit is elliptic. If $a < 0$ the orbit is hyperbolic.
h	The projection of the eccentricity vector onto the $\hat{\mathbf{y}}_{ep}$ axis.
k	The projection of the eccentricity vector onto the $\hat{\mathbf{x}}_{ep}$ axis.
p	The projection of $\mathbf{N}$ onto the $\hat{\mathbf{y}}_{ep}$ axis.
q	The projection of $\mathbf{N}$ onto the $\hat{\mathbf{x}}_{ep}$ axis.
λ	The mean longitude.

Table 1.4: The Spherical Elements

Symbol	Name	Description
r	r	Magnitude of the position vector, $\|\mathbf{r}\|$
λ	Right Ascension	The between the projection of $\mathbf{r}$ into the $x-y$ and the x-axis. Measured counterclockwise.
δ	Declination	The angle between $\mathbf{r}$ and the $x-y$ plane measured in the plane formed by $\mathbf{r}$ and $\hat{\mathbf{z}}$.
v	v	Magnitude of the velocity vector, $\|\mathbf{v}\|$
ψ	Flight path angle	The angle measured from a plane normal to $\mathbf{r}$ to the velocity vector $\mathbf{v}$, measured in the plane formed by $\mathbf{r}$ and $\mathbf{v}$
α_f	Flight path azimuth	The angle measured from vector perpendicular $\mathbf{r}$ and pointing north, to the projection of $\mathbf{v}$ into a plane normal to $\mathbf{r}$.

Table 1.5: The Modified Keplerian Elements

Symbol	Name	Description
r_p	radius of periapsis	The radius of periapsis is the radius at the spacecrafts closest approach to the central body. The radius of periapsis must be greater than zero, parabolic orbits are not currently supported.
r_a	radius of apoapsis	For an elliptic orbit r_a is the radius at the spacecrafts farthest distance from the central body and $r_a > r_p$. For hyperbolic orbits, $r_a < r_p$ and $r_a < 0$
i	inclination	The inclination is the angle between the $\hat{\mathbf{z}}_I$ axis and the orbit normal direction $\mathbf{h}$. If $i \leq 90°$ then the orbit is prograde. If $i > 90°$ then the orbit is retrograde.
ω	argument of periapsis	The argument of periapsis is the angle between a vector pointing at periapsis, $\mathbf{x}_p$, and a vector pointing at the spacecraft. The argument of periapsis is undefined for circular orbits.
Ω	right ascension of the ascending node	Ω is defined as the angle between $\hat{\mathbf{x}}_I$ and $\mathbf{N}$ measured counterclockwise. $\mathbf{N}$ is defined as the vector pointing from the center of the central body to the spacecraft, when the spacecraft crosses the bodies equatorial plane from the southern to the northern hemisphere. Ω is undefined for equatorial orbits.
ν	true anomaly	The true anomaly is defined as the angle between a vector pointing at periapsis, $\mathbf{x}_p$, and a vector pointing at the spacecraft. The true anomaly is undefined for circular orbits.

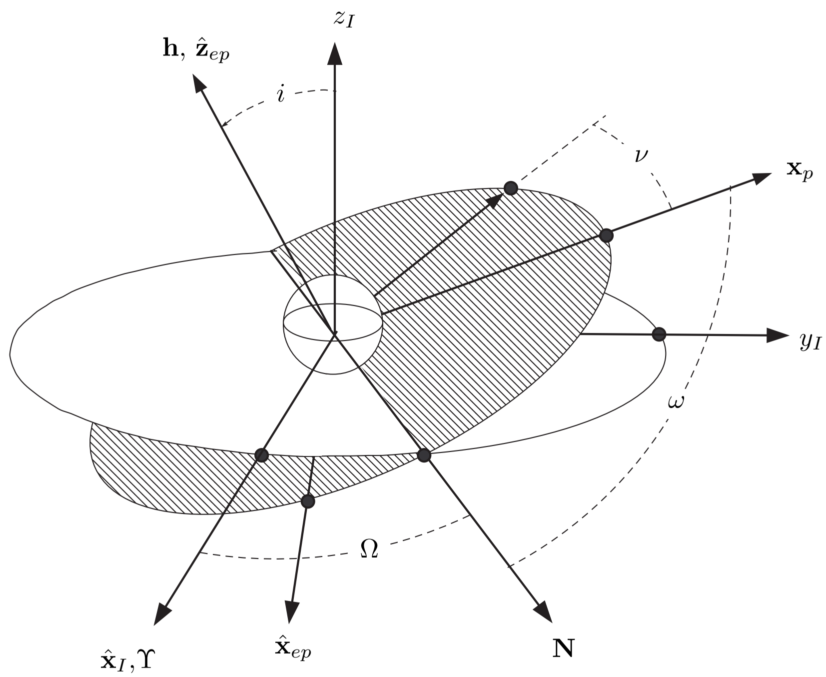

Figure 1.1: The Keplerian Elements

$$e = \sqrt{\left|1 - \frac{p}{a}\right|} \tag{1.8}$$

$$i = \cos^{-1}(\hat{h}_z) \tag{1.9}$$

There are three special cases that are considered when calculating the true anomaly. The first is for elliptic orbits when $e \geq 1e-6$ GMAT calculates $\cos\nu$ and $\sin\nu$ using the following two equations

$$\cos\nu = \frac{p-r}{er} \tag{1.10}$$

$$\sin\nu = \frac{\mathbf{r}\cdot\mathbf{v}}{\mu er} \tag{1.11}$$

The next two special cases are for circular orbits. For both cases, when $e < 1e-6$, GMAT sets the eccentricity to be identically zero.

$$e = 0.0 \quad \text{ if } \; e < 1e-6 \tag{1.12}$$

The second special case is the circular inclined case. In this case $e < 1e-6$ and $(i \geq 1e-6)$. For the circular inclined case $\cos\nu$ and $\sin\nu$ are calculated as described below.

$$\mathbf{N} = \begin{bmatrix} -\hat{h}_z\hat{h}_x & -\hat{h}_z\hat{h}_y & \hat{h}_x^2 + \hat{h}_y^2 \end{bmatrix}^T \tag{1.13}$$

where $\mathbf{N}$ is a unit vector pointing from the origin to the ascending node.

$$\cos\nu = \frac{-\hat{h}_z\hat{h}_x x - \hat{h}_z\hat{h}_y y}{r\left(\hat{h}_x^2 + \hat{h}_y^2\right)} \tag{1.14}$$

$$\sin\nu = \frac{-\hat{h}_z\hat{h}_x x - \hat{h}_z\hat{h}_y y + \left(\hat{h}_x^2 + \hat{h}_y^2\right) z}{rN} \tag{1.15}$$

The third special case for calculating ν is the the circular equatorial case. This occurs when $e < 1e-6$ and $(i < 1e-6)$

$$\cos\nu = \frac{x}{r}; \tag{1.16}$$

$$\sin\nu = \frac{y}{r}; \tag{1.17}$$

Finally, knowing $\cos\nu$ and $\sin\nu$ we can solve for ν using

$$\nu = \text{atan2}(\sin\nu, \cos\nu) \tag{1.18}$$

If ν is negative it is adjusted to fall between 0° and 360°.

Now GMAT calculates Ω. If $\mid i \mid < 1e-6$ then GMAT sets Ω to be identically zero.

$$\Omega = 0.0 \quad \text{ if } \; \mid i \mid < 1e-6 \tag{1.19}$$

Otherwise, Ω is calculated using

$$\Omega = \text{atan2}(\hat{h}_x, -\hat{h}_y) \tag{1.20}$$

If Ω is negative it is adjusted to fall between 0° and 360°.

The last of the orbital elements to be calculated by GMAT is ω. There are several special cases. For near circular, or circular orbits when $e < 1e-6$, ω is set to be identically zero.

$$\omega = 0.0 \quad \text{ if } \; e < 1e-6 \tag{1.21}$$

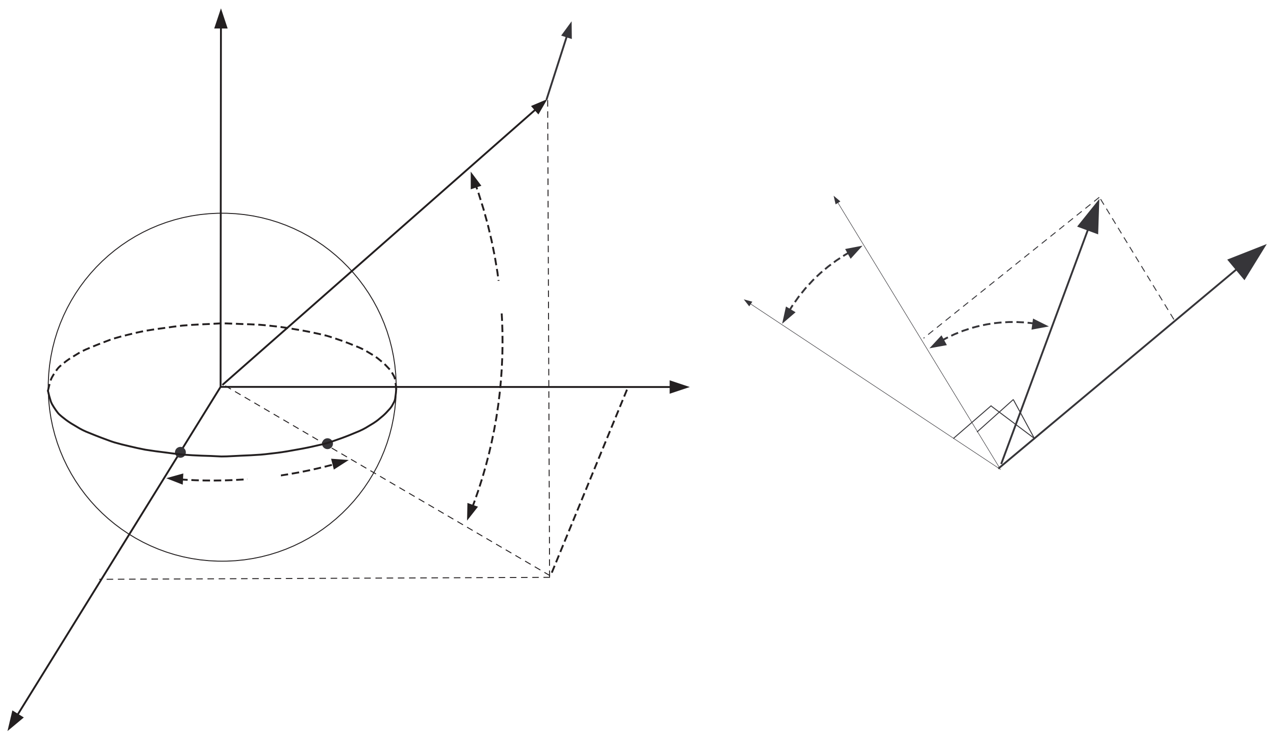

Figure 1.2: The Spherical Elements

The next two special cases are for elliptical orbits. Both require $\mathbf{X}$ which is defined as

$$\mathbf{X} = \left(v^2 - \frac{\mu}{r}\right)\mathbf{r} - (\mathbf{r}\cdot\mathbf{v})\,\mathbf{v} \tag{1.22}$$

For elliptical inclined orbits, when $e \geq 1e-6$ and $\mid i \mid \geq 1e-6$, ω is calculated using

$$\omega = \cos^{-1}\frac{\mathbf{N}\cdot\mathbf{X}}{NX} \tag{1.23}$$

If $e \geq 1e-6$ and $\mid i \mid < 1e-6$ the ω is calculated as using

$$\omega = \text{atan2}(X_1, X_0) \tag{1.24}$$

If ω is negative it is adjusted to fall between $0°$ and $360°$.

Comments: The denominators in the following equations are checked before performing division: (1.7), (1.8), (1.10), (1.11) and (1.22). If a denominator is smaller than 10^{-30}, then the division is not performed and the function returns an error message.

1.1.3 Keplerian State to Cartesian State

if $i < 0$ then i = i+180

if $i <$ 1e-6 then $\Omega = 0$

if $e <$ 1e-6 then $\omega = 0$

$$p = a(1-e^2); \tag{1.25}$$

if $(1 + e\cos\nu <$ 1e-30) then error and return, else

$$r = \frac{p}{1+e\cos\nu} \tag{1.26}$$

$$x = r\left(\cos\left(\omega+\nu\right)\cos\Omega - \cos i\sin\left(\omega+\nu\right)\sin\Omega\right) \tag{1.27}$$

$$y = r\left(\cos\left(\omega+\nu\right)\sin\Omega + \cos i\sin\left(\omega+\nu\right)\cos\Omega\right) \tag{1.28}$$

$$z = r\left(\sin\left(\omega+\nu\right)\sin i\right) \tag{1.29}$$

if $(\|p\| < 1e-30)$ then error and return, else

$$\dot{x} = \sqrt{\frac{\mu}{p}}\left[(\cos\nu + e)\left(-\sin\omega\cos\Omega - \cos i\sin\Omega\cos\omega\right) - \sin\nu\left(\cos\omega\cos\right.\right. \tag{1.30}$$

$$\dot{y} = \sqrt{\frac{\mu}{p}}\left[(\cos\nu + e)\left(-\sin\omega\sin\Omega + \cos i\cos\Omega\cos\omega\right) - \sin\nu\left(\cos\omega\sin\right.\right. \tag{1.31}$$

$$\dot{z} = \sqrt{\frac{\mu}{p}}\left[(\cos\nu + e)\sin i\cos w - \sin\nu\sin i\sin\omega\right] \tag{1.32}$$

1.1.4 Equinoctial to Cartesian

The equinoctial elements used in GMAT are defined as follows

a = semimajor axis

h = projection of the eccentricity vector **e**

$$\lambda = F + h\cos F - k\sin F \tag{1.33}$$

$$\beta = \frac{1}{a + \sqrt{1 - h^2 - k^2}} \tag{1.34}$$

$$X_1 = a\left[(1 - h^2\beta)\cos F + hk\beta\sin F - k\right] \tag{1.35}$$

$$Y_1 = a\left[(1 - k^2\beta)\sin F + hk\beta\cos F - k\right] \tag{1.36}$$

$$\dot{X}_1 = \frac{na^2}{r}\left[hk\beta\cos F - (1 - h^2\beta)\sin F\right] \tag{1.37}$$

$$\dot{Y}_1 = \frac{na^2}{r}\left[(1 - k^2\beta)\cos F - hk\beta\sin F\right] \tag{1.38}$$

The transformation from the equinoctial system to the inertial Cartesian system is given by

$$\mathbf{r} = X_1\hat{\mathbf{f}} + Y_1\hat{\mathbf{g}} \tag{1.39}$$

$$\mathbf{v} = \dot{X}_1\hat{\mathbf{f}} + \dot{Y}_1\hat{\mathbf{g}} \tag{1.40}$$

where

$$\begin{bmatrix}\hat{\mathbf{f}} & \hat{\mathbf{g}} & \hat{\mathbf{w}}\end{bmatrix} = \frac{1}{1 + p^2 + q^2}\begin{pmatrix} 1 - p^2 + q^2 & 2pqj & 2p \\ 2pq & \left(1 + p^2 - q^2\right) & -2q \\ -2pj & 2q & \left(1 - p^2 - q^2\right)j \end{pmatrix} \tag{1.41}$$

where

j = 1 for direct orbits ($0 \le i \le 90°$)
j = -1 for retrograde orbits ($90 < i \le 180°$)

1.1.5 Cartesian to Equinoctial

$$a = \frac{1}{\frac{2}{r} - \frac{v^2}{\mu}} \tag{1.42}$$

$$\mathbf{e} = -\frac{\mathbf{v}}{r} - \frac{(\mathbf{r}\times\mathbf{v})\times\mathbf{v}}{\mu} \tag{1.43}$$

$$\hat{\mathbf{h}} = \frac{\mathbf{r}\times\mathbf{v}}{\|\mathbf{r}\times\mathbf{v}\|} \tag{1.44}$$

$$f_x = 1 - \frac{w_x^2}{1 + w_z^j} \tag{1.45}$$

$$f_y = -\frac{w_x w_y}{1 + w_z^j} \tag{1.46}$$

$$f_z = -w_x^j \tag{1.47}$$

where j is described in Section 1.1.4.

$$\hat{\mathbf{g}} = \hat{\mathbf{w}} \times \hat{\mathbf{f}} \tag{1.48}$$

the elements h, k, p, and q are given by

$$h = \mathbf{e}\cdot\hat{\mathbf{g}} \tag{1.49}$$

$$k = \mathbf{e}\cdot\hat{\mathbf{f}} \tag{1.50}$$

$$p = \frac{w_x}{1 + w_z^j} \tag{1.51}$$

$$q = -\frac{w_y}{1 + w_z^j} \tag{1.52}$$

The mean longitude, λ, is computed using the generalized Kepler equation

$$\lambda = F + h\cos f - k\sin F \tag{1.53}$$

where

$$F = \tan^{-1}\left(\frac{\sin F}{\cos F}\right) \tag{1.54}$$

with

$$\cos F = k + \frac{\left(1 - k^2\beta\right)X_1 - hk\beta Y_1}{a\sqrt{1 - h^2 - k^2}} \tag{1.55}$$

$$\sin F = h + \frac{\left(1 - h^2\beta\right)Y_1 - hk\beta X_1}{a\sqrt{1 - h^2 - k^2}} \tag{1.56}$$

The parameter β is given by Eq. 1.34. Finally, the position coordinates x_{ep} and y_{ep} relative to the equinoctial coordinate system are given by

$$X_1 = \mathbf{r}\cdot\hat{\mathbf{f}} \tag{1.57}$$

$$Y_1 = \mathbf{r}\cdot\hat{\mathbf{g}} \tag{1.58}$$

1.2 Cartesian to Spherical

1.3 Spherical to Cartesian

1.4 Keplerian to Modified Keplerian

Given: a, e, i, ω, Ω, and ν

Find: r_p and r_a

$$r_p = a(1 - e); \tag{1.59}$$

$$r_a = a(1 + e); \tag{1.60}$$

1.5 Modified Keplerian to Keplerian

Given: r_p, r_a, i, ω, Ω, and ν

Find: a and e

$$e = \frac{1 - \frac{r_p}{r_a}}{1 + \frac{r_p}{r_a}} \tag{1.61}$$

$$a = \frac{r_p}{1 - e} \tag{1.62}$$

1.6 RadApo

Given: a, and e

Find: r_a

If e > (1 - 1e^{-12}) then $r_a = 0$.

Otherwise,

$$r_a = a(1 + e) \tag{1.63}$$

Comment: a and e are calculated from the satellite cartesian state as shown in Section 1.1.2.

1.7 RadPer

Given: a, and e

Find: r_p

$$r_p = a(1 - e) \tag{1.64}$$

Comment: a and e are calculated from the satellite cartesian state as shown in Section 1.1.2.

1.8 VelApoapsis

Given: a, e, and μ

Find: v_a

If e > (1 - 1e^{-12}) then $v_a = 0$.

Otherwise,

$$v_a = \sqrt{\frac{\mu}{a}\left(\frac{1 - e}{1 + e}\right)} \tag{1.65}$$

Comment: a and e are calculated from the satellite cartesian state as shown in Section 1.1.2, and μ is associated with the specified central body.

1.9 VelPeriapsis

Given: a, e, and μ

Find: v_p

$$v_a = \sqrt{\frac{\mu}{a}\left(\frac{1 + e}{1 - e}\right)} \tag{1.66}$$

Comment: a and e are calculated from the satellite cartesian state as shown in Section 1.1.2, and μ is associated with the specified central body.

1.10 Beta Angle

Definition: The Beta angle, β, is defined as the angle between the orbit normal vector, and the vector from the celestial body to the sun.

$$\hat{\mathbf{h}} = \frac{\mathbf{r}_\oplus \times \mathbf{v}_\oplus}{\|\mathbf{r}_\oplus \times \mathbf{v}_\oplus\|}$$

$$\hat{\mathbf{r}}_{s\oplus} = \frac{\mathbf{r}_{s\oplus}}{\|_{s\oplus}\|}$$

$$\beta = \cos^{-1}\left(\hat{\mathbf{h}} \cdot \hat{\mathbf{r}}_{s\oplus}\right)$$

- $\mathbf{r}_\oplus$: Position vector of spacecraft with respect to celestial body, in the EarthMJ2000Eq system.
- $\mathbf{v}_\oplus$: Velocity vector of spacecraft with respect to celestial body, in the EarthMJ2000Eq system.
- $\mathbf{r}_{s\oplus}$: Position vector from celestial body, to the sun.

1.11 B-Plane Coordinates

Given: $\mathbf{r}$, $\mathbf{v}$, and selected coordinate system

Find: B_R and B_T

This method was adopted from work by Kizner.[5] If the selected coordinate system does not have a celestial body as its origin, then exit and throw an error message. GMAT will not support this currently.

Step 1: Convert $\mathbf{r}$ and $\mathbf{v}$ to selected coordinate system if they are not already in that system.

$$r = \|\mathbf{r}\|$$

$$v = \|\mathbf{v}\|$$

Calculate eccentricity related information

$$\mathbf{e} = \frac{\left(v^2 - \frac{\mu}{r}\right)\mathbf{r} - (\mathbf{r}\cdot\mathbf{v})\mathbf{v}}{\mu}$$

$$e = \|\mathbf{e}\|$$

$$\hat{\mathbf{e}} = \frac{\mathbf{e}}{e}$$

If $e \leq 1$, then the method fails and returns.

Now let's calculate the angular momentum and orbit normal vectors.

$$\mathbf{h} = \mathbf{r}\times\mathbf{v}$$

$$h = \|\mathbf{r}\times\mathbf{v}\|$$

$$\hat{\mathbf{h}} = \frac{\mathbf{h}}{h}$$

A unit vector normal to both the eccentricity vector and the orbit normal vector is defined as:

$$\hat{\mathbf{n}} = \hat{\mathbf{h}}\times\hat{\mathbf{e}}$$

The following relations are only true for hyperbolic orbits: The semiminor axis, b, can be calculated using

$$b = \frac{h^2}{\mu\sqrt{e^2-1}}$$

The incoming asymptote is defined using

$$\mathbf{S} = \frac{\hat{\mathbf{e}}}{e} + \sqrt{1-\left(\frac{1}{e}\right)^2}\,\hat{\mathbf{n}}$$

The B-vector, $\mathbf{B}$, is calculated using

$$\mathbf{B} = b\left(\sqrt{1-\left(\frac{1}{e}\right)^2}\,\hat{\mathbf{e}} - \frac{1}{e}\hat{\mathbf{n}}\right)$$

The remaining vectors, $\mathbf{T}$ and $\mathbf{R}$ are found using

$$\mathbf{T} = \frac{[\,S_y \quad -S_x \quad 0\,]^T}{\sqrt{S_x^2+S_y^2}}$$

$$\mathbf{R} = \mathbf{S}\times\mathbf{T}$$

Finally, the desired quantities are found using

$$B_T = \mathbf{B}\cdot\mathbf{T}$$

$$B_R = \mathbf{B}\cdot\mathbf{R}$$

1.12 Magnitude and Angle of the B Vector

To avoid code reduplication, the magnitude and angle of the B vector, $\|\mathbf{B}\|$ and θ_B respectively, are calculated from the outputs of the B-Plane coordinates algorithm. The equations for $\|\mathbf{B}\|$ and θ_B are

$$\|\mathbf{B}\| = \sqrt{B_T^2 + B_R^2} \tag{1.67}$$

$$\theta_B = \tan^{-1}\frac{B_R}{B_T} \tag{1.68}$$

which is implemented using $atan2(B_R, B_T)$

1.13 Mean Anomaly

Given: ν, e

Find: M

If e < (1 - 1e^{-12}) then calculate E using algorithm in Sec. 1.14. Then M is calculated using

$$M = E - e\sin E \tag{1.69}$$

Note: E must be expressed in radians in the above equation, and results in M in radians.

If e > (1 + 1e^{-12}) then calculate H using algorithm in Sec. 1.15. Then M is calculated using

$$M = e\sinh H - H \tag{1.70}$$

Note: H must be expressed in radians in the above equation, and results in M in radians.

If neither of the above conditions are satisfied, $M = 0$, and output "Warning: Orbit is near parabolic in mean anomaly calculation. Setting MA = 0".

1.14 Eccentric Anomaly

Given: ν, e

Find: E

If e > (1 - 1e^{-12}) then $E = 0$, return.

Otherwise,

$$\sin(E) = \frac{\sqrt{1-e^2}\sin(\nu)}{1+e\cos\nu} \quad (1.71)$$
$$\cos(E) = \frac{e+\cos\nu}{1+e\cos\nu} \quad (1.72)$$
$$E = \text{atan2}(\sin E, \cos E) \quad (1.73)$$

1.15 Hyperbolic Anomaly

Given: ν, e

Find: H

If e < (1 + 1e^{-12}) then $H = 0$, return.

Otherwise,

$$\sinh(H) = \frac{\sin(\nu)\sqrt{e^2-1}}{1+e\cos\nu} \quad (1.74)$$
$$\cosh(H) = \frac{e+\cos\nu}{1+e\cos\nu} \quad (1.75)$$
$$H = \tanh^{-1}(\frac{\sinh H}{\cosh H}) \quad (1.76)$$

1.16 Orbit Period

Given: a, and μ

Find: T

If a < 0, then $T = 0$, return.

Otherwise,

$$T = 2\pi\sqrt{\frac{a^3}{\mu}} \quad (1.77)$$

Comment: a is calculated from the satellite cartesian state as shown in Section 1.1.2, and μ is associated with the specified central body.

1.17 C3Energy

Given: a, and μ

Find: C_3

$$C_3 = -\frac{\mu}{a} \quad (1.78)$$

Comment: a is calculated from the satellite cartesian state as shown in Section 1.1.2, and μ is associated with the specified central body.

1.18 Energy

Given: a, and μ

Find: $\mathcal{E}$

$$\mathcal{E} = -\frac{\mu}{2a} \quad (1.79)$$

Comment: a is calculated from the satellite cartesian state as shown in Section 1.1.2, and μ is associated with the specified central body.

1.19 Mean Motion

Given: a, e, and μ

Find: n

The orbit is considered elliptic if $e > 1 + 1e^{-10}$. In this case the mean motion, n, is calculated using

$$n = \sqrt{\frac{\mu}{a^3}} \quad (1.80)$$

The orbit is considered hyperbolic if $e > 1 - 1e^{-10}$. In this case the mean motion, n, is calculated using

$$n = \sqrt{-\frac{\mu}{a^3}} \quad (1.81)$$

If neither of the above two conditions are met, the mean motion is calculated using

$$n = 2\sqrt{\mu} \quad (1.82)$$

Comment: a and e are calculated from the satellite cartesian state as shown in Section 1.1.2, and μ is associated with the specified central body.

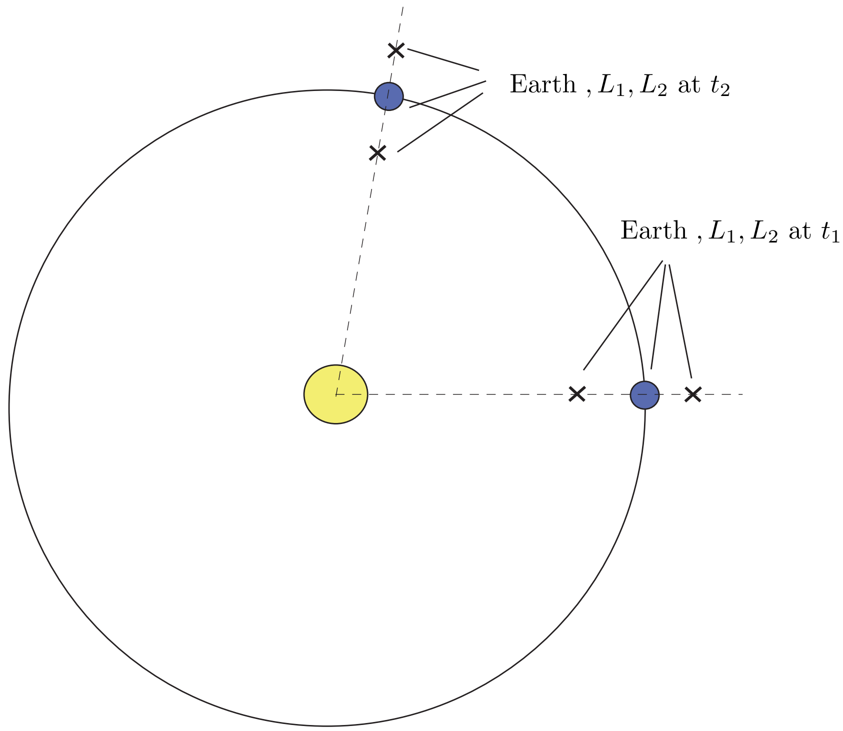

Figure 1.3: Description of Libration Points

1.20 Semilatus Rectum

1.21 Time Calculations

1.21.1 Elapsed Days

1.21.2 Hour Angle

1.22 Libration Points

We begin by assuming that the planets move in circular orbits about the sun, and the mass of a spacecraft is negligible compared to the mass of the planets. For illustrative purposes, lets consider the Earth and its orbit about the Sun. In this case, the libration points are locations in space where a spacecraft will stay fixed with respect to the Earth and Sun. Figure 1.3 shows a simple illustration. We see the Sun, the Earth's position with respect to the Sun, and the Libration points L_1 and L_2 at two different epochs. Notice that at t_1, the points L_1 and L_2 are on the Earth-Sun line. At a later time, t_2, although the Earth has moved with respect to the sun, L_1 and L_2 still lie on the Earth-Sun line.

The preceding example gives a brief qualitative description of two of the Earth-Sun libration points. In general, there are five libration points for a given three body system. To determine the locations of the libration points, it is convenient to work in a rotating coordinate system rather than the inertial system shown in Fig. 1.3. The system we use is constructed as follows:

- Define the primary as the heavier of the two bodies, the secondary as the lighter.
- Define the coordinate system x-axis as the axis pointing from the primary to the secondary.
- Define the y-axis to be orthogonal to the x-axis in the plane of the secondary's motion about the primary, pointing in the direction the secondary moves about the primary.
- Define the z-axis orthogonal to the x and y axes to form a right-handed system.
- Place the origin at center-of-mass of the system.

This coordinates system is illustrated in Fig. 1.4. The locations of the libration points in the rotating coordinate system can be found by calculating the values of γ that solve the following equations:

$$\gamma_1^5 - (3-\mu^*)\,\gamma_1^4 + (3-2\mu^*)\,\gamma_1^3 - \mu^*\gamma_1^2 + 2\mu^*\gamma_1 - \ldots$$
$$\gamma_2^5 + (3-\mu^*)\,\gamma_2^4 + (3-2\mu^*)\,\gamma_2^3 - \mu^*\gamma_2^2 - 2\mu^*\gamma_2 - \ldots$$
$$\gamma_3^5 + (2+\mu^*)\,\gamma_3^4 + (1+2\mu^*)\,\gamma_3^3 - (1-\mu^*)\,\gamma_3^2 - 2\,(1-\mu^*)\,\gamma_3 - (1-\mu\ldots$$

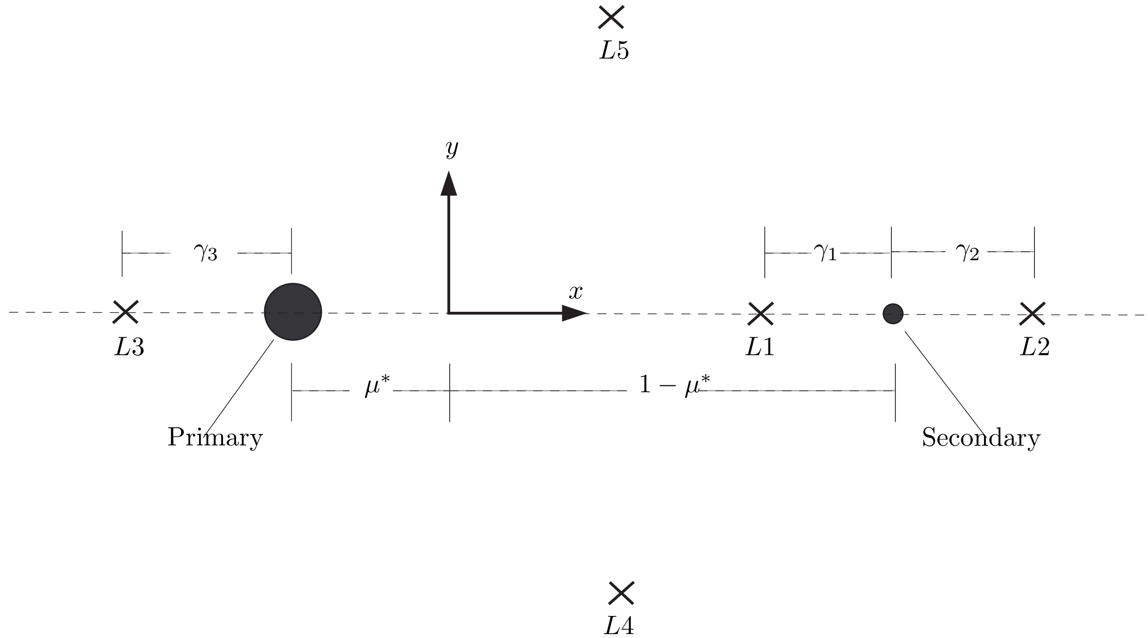

Figure 1.4: Definition of Libration Points

where

$$\mu^* = \frac{m_2}{m_1 + m_2} \tag{1.86}$$

Equations (1.83)-(1.85) do not have exact analytic solutions. Szebehely[6] presents series solutions for γ_1 and γ_2, and γ_3. The expansions for γ_1 and γ_2 are:

$$\gamma_1 = \left(\frac{\mu}{3}\right)^{1/3} - \frac{1}{3}\left(\frac{\mu}{3}\right)^{2/3} - \frac{1}{9}\left(\frac{\mu}{3}\right)^{1} - \frac{23}{81}\left(\frac{\mu}{3}\right)^{4/3} + \frac{151}{243}\left(\frac{\mu}{3}\right)^{5/3} - \frac{1}{9}\left(\frac{\mu}{3}\right)^{2} \tag{1.87}$$

$$\gamma_2 = \left(\frac{\mu}{3}\right)^{1/3} + \frac{1}{3}\left(\frac{\mu}{3}\right)^{2/3} - \frac{1}{9}\left(\frac{\mu}{3}\right)^{1} - \frac{31}{81}\left(\frac{\mu}{3}\right)^{4/3} - \frac{119}{243}\left(\frac{\mu}{3}\right)^{5/3} - \frac{1}{9}\left(\frac{\mu}{3}\right)^{2} \tag{1.88}$$

However, Szebehely[6] notes that Eqs. (1.83)-(1.85) are most easily solved using an iterative method with the following as the initial guesses:

$$\gamma_1 = \gamma_2 = \left(\frac{\mu^*}{3\left(1-\mu^*\right)}\right)^{1/3} \tag{1.89}$$

$$\gamma_3 = 1 \tag{1.90}$$

GMAT uses the Newton-Raphson method to solve for the roots of the equations by iterating on

$$\gamma(i+1) = \gamma(i) - \frac{F(\gamma(i))}{F'(\gamma(i))} \tag{1.91}$$

until the the difference $|\gamma(i+1) - \gamma(i)| < 10^{-8}$. The derivative $F'(\gamma)$ for each libration point is shown below.

$$\text{(For L1)} \quad F'(\gamma) = 5\gamma_1^4 - 4\left(3-\mu^*\right)\gamma_1^3 + 3\left(3-2\mu^*\right)\gamma_1^2 - 2\mu^*\gamma_1 + 2\mu^* \tag{1.92}$$

$$\text{(For L2)} \quad F'(\gamma) = 5\gamma_2^4 + 4\left(3-\mu^*\right)\gamma_2^3 + 3\left(3-2\mu^*\right)\gamma_2^2 - 2\mu^*\gamma_2 - 2\mu^* \tag{1.93}$$

$$\text{(For L3)} \quad F'(\gamma) = 5\gamma_3^4 + 4\left(2+\mu^*\right)\gamma_3^3 + 3\left(1+2\mu^*\right)\gamma_3^2 - 2\left(1-\mu^*\right)\gamma_3 - 2\left(1-\mu^*\right) \tag{1.94}$$

Table 1.6: Location of Libration Points in RLP Frame, with the Origin at the Primary Body

Point	x-Position	y-Position
L1	$1-\gamma_1$	0
L2	$1+\gamma_2$	0
L3	$-\gamma_3$	0
L4	1/2	$\sqrt{3}/2$
L5	1/2	-$\sqrt{3}/2$

We now need to redimensionalize the results found in the rotating system, and perform the necessary transformations to obtain the results in the MJ2000 system. Let's assume that $\mathbf{r}_s$, $\mathbf{v}_s$, and $\mathbf{a}_s$ are the position, velocity, and acceleration vectors respectively of the secondary body, with respect to the primary body, expressed in the FK5 system. Then, the position of the i^{th} libration point can be expressed in the rotating system with the origin centered on the primary body as

$$\mathbf{r}^i = r_s \left[x_i \quad y_i \quad 0\right]^T \tag{1.95}$$

where

$$r_s = \|\mathbf{r}_s\| \tag{1.96}$$

The velocity of the i^{th} libration point can be expressed in the rotating system with the origin centered on the primary body as

$$\mathbf{v}^i = \frac{\mathbf{v}_s \cdot \mathbf{r}_s}{r_s}\left[x_i \quad y_i \quad 0\right]^T \tag{1.97}$$

Now we have the redimensionalized position and velocity vectors of the libration point in the rotating coordinate system defined by the motion of the secondary body with respect to the primary body. To determine the position and velocity vectors in the FK5 system, with the origin located at the primary body, we need to determine the rotation matrix and its derivative as follows:

$$\mathbf{R}^{Ii} = \begin{pmatrix} \hat{x}_1 & \hat{y}_1 & \hat{z}_1 \\ \hat{x}_2 & \hat{y}_2 & \hat{z}_2 \\ \hat{x}_3 & \hat{y}_3 & \hat{z}_3 \end{pmatrix} \tag{1.98}$$

and

$$\dot{\mathbf{R}}^{Ii} = \begin{pmatrix} \dot{\hat{x}}_1 & \dot{\hat{y}}_1 & \dot{\hat{z}}_1 \\ \dot{\hat{x}}_2 & \dot{\hat{y}}_2 & \dot{\hat{z}}_2 \\ \dot{\hat{x}}_3 & \dot{\hat{y}}_3 & \dot{\hat{z}}_3 \end{pmatrix} \tag{1.99}$$

where

$$\hat{\mathbf{x}} = \frac{\mathbf{r}_s}{r_s} \tag{1.100}$$

$$\hat{\mathbf{z}} = \frac{\mathbf{r}_s \times \mathbf{v}_s}{\|\mathbf{r}_s \times \mathbf{v}_s\|} \tag{1.101}$$

$$\hat{\mathbf{y}} = \hat{\mathbf{z}} \times \hat{\mathbf{x}} \tag{1.102}$$

and

$$\dot{\hat{\mathbf{x}}} = \dot{\hat{\mathbf{r}}}_s = \frac{\mathbf{v}_s}{r_s} - \frac{\hat{\mathbf{r}}_s}{r_s}\left(\hat{\mathbf{r}}_s \cdot \mathbf{v}_s\right) \tag{1.103}$$

$$\dot{\hat{\mathbf{z}}} = \frac{\mathbf{r}_s \times \mathbf{a}_s}{\|\mathbf{r}_s \times \mathbf{v}_s\|} - \frac{\hat{\mathbf{z}}}{\|\mathbf{r}_s \times \mathbf{v}_s\|}\left(\mathbf{r}_s \times \mathbf{a}_s \cdot \hat{\mathbf{z}}\right) \tag{1.104}$$

$$\dot{\hat{\mathbf{y}}} = \dot{\hat{\mathbf{z}}} \times \hat{\mathbf{x}} + \hat{\mathbf{z}} \times \dot{\hat{\mathbf{x}}} \tag{1.105}$$

GMAT currently assumes that the terms $\mathbf{r}_s \times \mathbf{a}_s$ are zero.

We finally arrive at the position of the Libration Point in the FK5 system with the origin at the primary by performing the calculations:

$$\mathbf{r} = \mathbf{R}^{Ii}\mathbf{r}^i \tag{1.106}$$

$$\mathbf{v} = \dot{\mathbf{R}}^{Ii}\mathbf{r}^i + \mathbf{R}^{Ii}\mathbf{v}^i \tag{1.107}$$

Note that in the GMAT script language, we can define this coordinate system using

```
Create CoordinateSystem CS;
CS.Origin    = PrimaryBodyName;
CS.Primary   = PrimaryBodyName;
CS.Secondary = SecondaryBodyName;
CS.XAxis = R;
CS.ZAxis = N;
```

1.23 Barycenter

The barycenter of a system of point masses, $\mathbf{r}_b$, is also called the center of mass. If we have a system of n bodies, and we know the position of the i^{th} body with respect to a common reference system, then we can calculate the barycenter of the system using

$$\mathbf{r}_b = \frac{\sum_{i=1}^{n} m_i \mathbf{r}_i}{\sum_{i=1}^{n} m_i} \tag{1.108}$$

Similarly, we can calculate the velocity of the barycenter using the following equation

$$\mathbf{v}_b = \frac{\sum_{i=1}^{n} m_i \mathbf{v}_i}{\sum_{i=1}^{n} m_i} \tag{1.109}$$

1.24 Umbra and Penumbra

The Umbra and Penumbra functions are used to determine if a spacecraft is in the shadow of a celestial body. For both functions, if the value is less than 1, then the body is in shadow, if the function is greater than 1, then the body is not in shadow.

To calculate the Umbra or Penumbra function the user must first select a celestial body, and a spacecraft. GMAT calculates the spacecraft's position, in the coordinates system centered at the celestial body with the FK5 axis system. Next GMAT calculates the Sun's position in the same coordinate system.

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